International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Special Issue 2, July 2015, PP 575-579, ISSN 2347-307X (Print)
www.arcjournals.org

LEFT MULTIPLICATIVE GENERALIZED DERIVATIONS IN SEMI PRIME RINGS

C. Jaya Subba Reddy[1], V. Vijay kumar[2], S.MallikarjunaRao[3]
[1,2,3]Department of Mathematics, S.V.University, Tirupati,AndhraPradesh,India.

Abstract: *Let R be a ring. A map $F: R \to R$ is called left multiplicative generalized derivation if $F(xy) = d(x)y + xF(y)$ is fulfilled for all x, y in R.Where $d: R \to R$ is any map (not necessarily derivative or an additive mapping).The following results are proved:*

(i) $F(xy) \pm xy = 0$,
(ii) $F(xy) \pm yx = 0$,
(iii) $F(x)F(y) \pm xy = 0$,
(iv) $F(x)F(y) \pm yx = 0$ *, for all* $x, y \in S$.

Key words: *Semi prime ring, Multipicative generalized derivation,Left multiplicative generalized* derivation.

INTRODUCTION AND PRELIMINARIES

Let R be an associative ring. The center of R is denoted by Z. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. we shall make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if for $a, b \in R, aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for $a \in R, aRa = (0)$ implies $a = 0$. An additive map d from R to R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let $F: R \to R$ be a map associated with another map $g: R \to R$ so that $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$. If F is additive and g is a derivation of R, then F is said to be a generalized derivation of$R$ that was introduced by Breasar [3]. In [6], Hvala gave the algebraic study of generalized derivations of prime rings. The concept of multiplicative derivations appears for the first time in the work of Daif [4] and it was motivated by the work of Martindale[7]. The notion of multiplicative derivation was extended in Daifas follows. A map $F: R \to R$ is called a multiplicative generalized derivations if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In this definition we consider that d is any map (not necessarily a additive). To give its precise definition, we make a slight generalization of Daif and Tammam-El-Sayiad's definition for multiplicative generalizedderivation was extended Daif (see [5]).A map $F: R \to R$ (not necessarily additive) is called multiplicative generalized derivation if $F(xy) = F(x)\, y + x\, g(y)$forall $x, y \in R$, where g is any map (not necessarily derivation or additive map). A map $F: R \to R$ (not necessarily additive) is called left multiplicative generalized derivation if $F(xy) = g(x)\, y + x\, F(y)$ for all $x, y \in R$ where g is any map (not necessarily derivation or additive map).BasudebDhara and Shakir Ali [2] have studied multiplicative generalized derivations in prime rings and semi prime rings. In this paper we extended some results of left multiplicative generalized derivations in semi prime rings.

Main results

Theorem 1: Let R be a semi prime ring,S be a nonzero right ideal of the R and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $: R \to R$. If $F(xy) \pm xy = 0$ for all $x, y \in S$, then $g(S)S = (0), F(xy) = xF(y)$ for all $x, y \in S$ and F is a commuting map on S.

Proof: By the assumption, wehave $F(xy) - xy = 0$for all $x, y \in S$. (1)
Putting $x = zx$,$z \in S$ in (1) we get

$F(zxy) - zxy = 0$ for all $x, y, z \in S$. (2)
Since $F(xy) = g(x)y + xF(y)$ for all $x, y \in R$, it follows that,
$0 = g(z)xy + zF(xy) - zxy = g(z)xy + z(F(xy) - xy)$ for all $x, y, z \in S$. (3)
Application of (1) yields that
$g(z)xy = 0$ for all $x, y, z \in S$. (4)
Replacing $y = rg(z)y$, where $r \in R$, we get
$g(z)xrg(z)y = 0$
In particular, it follows that $g(z)xRg(z)x = 0$ for all $x, z \in S$.
Since R is a semiprime ring, the last expression forces that $g(z)x = 0$ for all $x, z \in S$, that is,

$$g(S)S = 0.$$

Thus, for any $x, y \in S$,

$$F(xy) = g(x)y + xF(y) = xF(y).$$

Then (1) implies that $x(F(y) - y) = 0$ for all $x, y \in S$.
This implies that $x(F(y) - y)Rx(F(y) - y) = (0)$.
Since R is semi prime ring, $x(F(y) - y) = 0$ for all $x, y \in S$.
Thus for any $x, y \in S$, $(F(y) - y)x = 0$ and $x(F(y) - y) = 0$ together implies that
$[x, (F(y) - y)] = 0$
$[x, F(y)] - [x, y] = 0$.
For $x = y$, we obtain $[x, F(x)] = 0$ for all $x \in S$.
That is, F is commuting on S.
In similar manner, we can prove that the same conclusion holds for $F(xy) + xy = 0$ for all $x, y \in S$.
Thereby the proof the theorem is completed.

Corollary 1: Let R be a semiprime ring and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm xy = 0$ for all $x, y \in R$, then $g = 0$ and $F(x) = \mp x$ (respectively) for all $x \in R$.

Proof: In view of theorem 1,we have $g = 0$, $F(xy) = xF(y)$ for all $x, y \in R$ and F is a commuting map on R.
Then by our hypothesis, we have for all $x, y \in R$.
That is $0 = xF(y) \pm xy = x(F(y) \pm y)$.
i.e. $R(F(y) - y) = (0)$.
Since R is semi prime ring, we conclude that $F(x) \pm x = 0$ for all $x \in R$.

Theorem 2: Let R be a semiprime ring , S be anon zero right ideal of R and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm yx = 0$ for all $x, y \in S$, then $[S, S]S = (0)$, $g(S)S = (0)$, $F(xy) = xF(y)$ for all $x, y \in S$ and F is commuting map on S.

Proof: First we consider the case $F(xy) - yx = 0$ for all $x, y \in S$. (5)
Putting $x = zx, z \in S$ in (5) we get

$$0 = F(zxy) - yzx = g(z)xy + zF(xy) - yzx = g(z)xy + z(F(xy) - yx) - yzx + zyx$$

$g(z)xy + z(F(xy) - yx) + (zy - yz)x = 0$ for all $x, y, z \in S$. (6)
From equation (5) &(6), we get $[z, y]x + g(z)xy = 0$. (7)
Now putting $z = y$ in equation (7) , we get $g(y)xy = 0$ for all $x, y \in S$.
Since S is right ideal of R, it follows that $g(y)xRy = (0)$
Which implies $g(y)yRg(y)y = (0)$ for all $y \in S$. Since R is semiprime, the last relation yields that $g(x)x = 0$ for all $x \in S$.
Substituting z for x in(7)and then using the fact that $g(x)x = 0$ for all $x \in S$, we obtain
$[x, y]x = 0$ for all $x, y \in S$.
Replace y by yz to get
$[x, yz]x = 0$
$[x, y]zx + y[x, z]x = 0$
$[x, y]zx = 0$ for all $x, y, z \in S$.
This implies $[x, y]z[x, y] = 0$ for all $x, y, z \in S$.
It follows that $[x, y]zR[x, y]z = (0)$ for all $x, y, z \in S$.
The semiprimeness of R forces that $[x, y]z = 0$ for all $x, y, z \in S$.
That is, $[S, S]S = (0)$.
Then (7) gives $g(z)xy = 0$ for all $x, y, z \in S$.
This implies $g(z)xRy = (0)$ and hence

$g(z)xRg(z)x = 0$for all $x, z \in S$.
Since R is semiprime, the above expression yields that $g(z)x = 0$ for all $x, z \in S$.
That is,$g(S)S = 0$.
Then for any $x, y \in S$, we have $F(xy) = g(x)y + xF(y) = xF(y)$.
Therefore , (5) becomes
$xF(y) - yx = 0$for all $x, y \in S$. (8)
Now putting $x = xy$ in (8) we obtain
$xyF(y) - yxy = 0$for all $x, y, z \in S$. (9)
Right multiplying (8) by y and then subtracting it from it from (9), we get
$x[F(y), y] = 0$forall $x, y \in S$. Since S is right ideal of R,we have
$xR[F(y), y] = (0)$for all $x, y \in S$. (10)
Replacing$x$ with y and then multiplying (10) from the left by $F(y)$ we find that
$F(y)yR[F(y), y] = (0)$for all $y \in S$. (11)
Replacing x by $yF(y)$ in (10) we get
$yF(y)R[F(y), y] = (0)$for all $x, y \in S$. (12)
Subtracting (11) from (12), we find that
$[F(y), y]R[F(y), y\} = (0)$ for all $y \in S$. (13)
The semi primeness of R gives that $[F(y), y] = 0$ for all $y \in S$.
$\Rightarrow [F(x), x] = 0$for all $x \in S$.
That is, F is commuting on S
Using a similar approach with necessary variations, we can prove that the same conclusion holds for the case $F(xy) + yx = 0$ for all $x, y \in S$.

Corollary 2: Let R be semiprime ring and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm yx = 0$ for all $x, y \in R$, then $g = 0, R$ is commutative and $F(x) = \mp x$(respect tvely) for all $x \in R$.

Proof: By the theorem 2, wehave$g = 0$,R is commutative, $F(xy) = xF(y)$ for all $x, y \in R$ and F is commuting map on R. Since R Is commutative, $F(xy) \pm yx = 0$becomes$(xy) \pm xy = 0$ for all $x, y \in R$. Then by Corollary 1, we conclude that $F(x) = \mp x$ (respectvely)for all $x \in R$.

Theorem 3: Let R be a semiprime ring, S be a nonzero right ideal of R and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(x)F(y) \pm xy = 0$ for all $x, y \in S$,then $g(S)S = 0, F(xy) = xF(y)$ for all $x, y \in S$,and$S[F(x), x] = 0$ for all $x \in S$.

Proof: Fist we assume that
$F(x)F(y) - xy = 0$for all $x, y \in S$. (14)
Substituting $x = zx$ in (14) and then using the given hypothesis, we find that
$F(zx)F(y) - zxy = 0$
$(g(z)x + zF(x))F(y) - zxy = 0$
$g(z)xF(y) + z(F(x)F(y) - xy) = 0$for all $x, y, z \in S$. (15)
Using (14), it reduces to $g(z)xF(y) = 0$ for all $x, y, z \in S$.
Replacing y by yu , $u \in S$ to get
$0 = g(z)xF(yu) = g(z)x(g(y)u + yF(u)$for all $x, y, z \in S$.
Using the fact that $g(z)xF(y) = 0$ for all $x, y, z \in S$,it gives
$g(z)xg(y)u = 0$ for all $x, y, z, u \in S$.
Since S is right ideal, it follows that$g(S)SRg(S)S = (0)$.
Since R is semiprime , we conclude that $g(S)S = (0)$.
Thus for any$x, y \in S$, we obtain $F(xy) = g(x)y + xF(y) = xF(y)$.
Replacing y by xy in (14), we get
$F(x)F(xy) - x^2y = 0$ (16)
That is, $F(x)xF(y) - x^2y = 0$for all $x, y, z \in S$. (17)
Left multiplying (14) by x and then subtracting it from (17), we get
$[F(x), x]F(y) = 0$for all $x, y \in S$.
Putting zy for y in the last relation, we obtain
$[F(x), x]zF(y) = 0$for all $x, y \in S$.
This implies $[F(x), x]z[F(x), x] = 0$

That is, $S[F(x),x]RS[F(x),x]=(0)$.
Since R is semiprime, it follows that

$$S[F(x),x]=(0) \text{ for all } x\in S.$$

A similar conclusion holds of the case $F(x)F(y)+xy=0$ for all $x,y,z\in S$.

Corollary 3: Let R be a semiprime ring and $F:R\to R$ be a left multiplicative generalized derivation associated with the map $g:R\to R$. If $F(x)F(y)\pm xy=0$ for all $x,y\in R$, $g=0$, $F(xy)=xF(y)$ for all $x,y\in R$and F is a commuting map on R.

Proof: By theorem 3, we conclude that $g=0$, $F(xy)=xF(y)$ for all $x,y\in R$, and F is a commuting map on R.

Theorem 4: Let R be a semiprime ring, S be a nonzero right ideal of R and $F:R\to R$ be a left multiplicative generalized derivation associated with the map $g:R\to R$. If $F(x)F(y)\pm yx=0$ for all $x,y\in S$, then $g(S)S=(0)$, $[S,S]S=(0)$, $F(xy)=xF(y)$ for all $x,y\in S$ and $[F(x),x]S=(0)$ for all $x\in S$.

Proof: First we assumed that

$F(x)F(y)-yx=0$for all $x,y\in S$. (18)

Replacing x with yx in (18), we obtain

$$0=F(yx)F(y)-y^2x=g(y)xF(y)+yF(x)F(y)-y^2x$$

$=g(y)xF(y)+y(F(x)F(y)-yx)$ (19)

Using (18), it reduces to $g(y)xF(y)=0$ for all $x,y\in S$.
Right multiplying by $F(z)$for $z\in S$,
$g(y)xF(y)F(z)=0$for all $x,y,z\in S$.
Using our hypothesis (18), the last expression yields$g(y)xzy=0$ for all $x,y,z\in S$ and hence
$g(y)SRSy=(0)$ for all $y\in S$.
Since R is semiprime, it must contain a family $P=\{P_\alpha\backslash\alpha\in\wedge\}$of prime ideals such $\cap P_\alpha=(0)$(se [1] for details). If P is atypical number of $P$and$x\in S$, it follows that
$Sx\subseteq P$ or $g(x)S\subseteq P$.
These two condition together imply that $g(x)Sx\subseteq P$ for any $P\in P$.
There fore ,$g(x)Sx\subseteq\cap_{\alpha\in\wedge}P_\alpha=(0)$.
That is,$g(x)Sx=0$ for all$x\in S$
Since S is a right ideal of R, it follows that $g(x)SRx=(0)$ for all $x\in S$.
Again, let P be the typical number of P and $x\in S$.
Then it follows that $x\in P$ or $g(x)S\subseteq P$
These two conditions together imply that $g(x)x\in P$ for all $x\in S$ and for any P ∈P .
Therefore $(x)Sx\subseteq\cap_{\alpha\in\wedge}P_\alpha=(0)$.
That is, $g(x)x=0$ for all $x\in S$.
Now by our hypothesis, we have for any $x,y,z\in S$ that

$$0=F(zx)F(y)-yzx=g(z)xF(y)+zF(x)F(y)-yzx$$

$0=g(z)xF(y)+z(F(x)F(y)-yx)+zyx-yzx=g(z)xF(y)+z(F(x)F(y)-yx)+(zy-yz)x=z(F(x)F(y)-yx)+[y,z]x+g(z)xF(y)=[y,z]x+g(z)xF(y)$

$=[y,z]x+g(z)xF(y)=0$ (20)

Replacing z with x, we obtain, by using $g(x)x=0$ for any $x\in S$, that
$[y,x]x=0$for any $x,y\in S$.
Then by the same argument as given in Theorem 2 we have $[S,S]S=(0)$
Then by (20), we have $g(z)xF(y)=0$ for all $x,y,z\in S$.
Now putting $y=zy$, we get $0=g(z)xF(zy)=g(z)xg(z)y+g(z)xzF(y)=g(z)xg(z)y=0$
Which yields$g(z)xRg(z)y=(0)$ for all $x,y,z\in S$.
Since R is a semiprime ring we get that $g(S)S=(0)$.
Hence, we conclude that $F(xy)=xF(y)$ for all $x,y\in S$.
Now in (18) replacing y with xy and then using the above fact, we get

$$F(x)F(xy)-xyx=0$$

$F(x)xF(y)-xyx=0$ for all $x,y\in S$. (21)

Now left multiplying (18) by x and then subtracting it from (21) we get $[F(x),x]F(y)=0$ for all $x,y\in S$.
Then replacing y by zy it yields $[F(x),x]zF(y)=0$ and hence

$$[F(x),x]Rz[F(y),y]=(0) \text{ for all } x,y,z\in S.$$

Since R is semiprime,$S[F(x),x]=(0)$ for all $x\in S$.

A similar conclusion holds for thecase $F(x)F(y) + yx = 0$ for all $x, y \in S$

Corollary 4: Let$R$ be semi prime ring and $F: R \to R$ be left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(x)F(y) \pm yx = 0$ for all $x, y \in R$,then $g = 0, R$ is commutatie, $F(xy) = xF(y)$ for all $x, y \in R$ and F is commuting on R.

Proof: By theorem 4 we get our conclusions.

REFERENCES

1. Anderson, F.w.: Letures on noncommutative rings. University of Oregon, Oregon (2002)
2. BasudebDhara and Shakir Ali:On multiplicative (generalized)-derivations in prime and semiprime rings Aequat. Math.86,56-79 (2013)
3. Bresar, M.: On the distance of the composition of two derivations to the generalized derivations. Glasgowmath.J.(33), 89-93 (1991)
4. Daif, M.N.:When is a multiplicative derivation additive. Int. J. Math. Math.Sci.14(3),615-618(1991)
5. Daif, M.N.,Tammam EI-Sayiad, M.S.: Mulplicative generalized derivations which are additive.East-West J.Math.9(1), 31-37(1997)
6. Hvala,B.:Generalized derivations in rings.Comm.Alegbra 26(4),1147-1166(1998)
7. Martindale,W.S.III.:When are multiplicative maps additive .Proc.Amer.Math.Soc.21,695-698(1969)